

The nonlinear HSS-like iteration method for absolute value equations

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Abstract Recently Salkuyeh proposed the Picard-HSS iteration method to solve the absolute value equation (AVE). In this paper, we focus our attention to improve the performance of the Picard-HSS iterative method, and propose a nonlinear HSS-like iterative method to solve AVE. Compared to that the Picard-HSS is an inner-outer double-layer iterative scheme, the HSS-like iteration is a monolayer and the iterative vector could be updated timely. By introducing a smoothing approximate function, we give a theoretical proof for the convergence of the nonlinear HSS-like iteration method for solving AVE. Some numerical experiments are used to demonstrate the feasibility, robustness and effectiveness of the nonlinear HSS-like method.

Keywords Absolute value equation · HSS-like iteration · Fixed point iteration · Positive definite · Convergence · Smoothing approximate function

1 Introduction

We consider the absolute value equation (AVE) of the form

$$Ax - |x| = b, \quad A \in \mathbb{C}^{n \times n}, \quad \text{and} \quad x, b \in \mathbb{C}^n \quad (1)$$

where $|x|$ denotes the component-wise absolute value of vector x , i.e., $|x| = (|x_1|, |x_2|, \dots, |x_n|)^T$. A slightly more general form of the AVE

$$Ax - B|x| = b, \quad A \in \mathbb{C}^{m \times n}, \quad B \in \mathbb{C}^{m \times n}, \quad \text{and} \quad x, b \in \mathbb{C}^m \quad (2)$$

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was introduced in [14] and investigated in a more general context in [4, 9]. Recently, these problems have been investigated in the literature [10, 15, 9, 5, 12].

The AVE (1) arises in linear programs, quadratic programs, bimatrix games and other problems, which can all be reduced to a linear complementarity problem (LCP)[3,10], and the LCP is equivalent to the AVE (1). This implies that AVE is NP-hard in its general form[10,9,5]. Beside, if $B = 0$, then the generalized AVE (2) reduces to a system of linear equations $Ax = b$, which have many applications in scientific computation[10].

The main research contents about AVE include two aspects in recent years, one is the theoretical analysis, which focuses on the theorem of alternatives, various equivalent reformulations, and the existence and nonexistence of solutions; see [3,4,14,12].

And the other is how to solve the AVE. In the last decade, based on the fact that the LCP can be reduced to the AVE, which owns a very special and simple structure, a large variety of methods for solving AVE (1) can be found in the literature; See [7,8,10,15]. For example, a finite computational algorithm that is solved by a finite succession of linear programs (SLP) in [5], and a semismooth Newton method is proposed in [6], which largely shortens the computation time than the SLP method. Furthermore, a smoothing Newton algorithm was presented in [3], which was proved to be globally convergent and the convergence rate was quadratic under the condition that the singular values of A exceed 1. This condition was weaker than the one used in [6].

Recently, The Picard-HSS iteration method is proposed to solve AVE in [16], which is designed to solve weakly nonlinear systems[2] and its generalization is also paid attention [18,13]. The sufficient conditions to guarantee the convergence of this method and some numerical experiments are given to show the effectiveness of the method. However, the numbers of the inner HSS iterative steps are often problem-dependent and difficult to be determined in actual computations. Moreover, the iterative vector can not be updated timely. In this paper, we present the nonlinear HSS-like iterative method to overcome the defect mentioned above, which is designed originally for solving weakly nonlinear systems in [2].

The rest of this paper is organized as follows. In Section 2 we review the HSS and Picard-HSS iteration methods. In section 3, we devote to introduce the nonlinear HSS-like iteration method to solve AVE (1) and investigate its convergence properties. Numerical experiments are presented in Section 4, to shown the feasibility and effectiveness of the nonlinear HSS-like method. Finally, in Section 5 we draw some conclusions.

2 The HSS and Picard-HSS iterative methods

2.1 The HSS iterative methods

Let $A \in \mathbb{C}^{n \times n}$ be a non-Hermitian positive definite matrix, $B \in \mathbb{C}^{n \times n}$ be a zero matrix, The generalized AVE (2) reduced to the system of linear equations

$$Ax = b. \quad (3)$$

As any square matrix A possesses a Hermitian and skew-Hermitian splitting (HSS)

$$A = H + S, \quad H = \frac{1}{2}(A + A^H) \quad \text{and} \quad S = \frac{1}{2}(A - A^H). \quad (4)$$

Bai et al. [1] established the following HSS iteration method to solve positive definite system of linear equations (3).

Algorithm 1 The HSS iteration method.

Given an initial guess $x^{(0)} \in \mathbb{C}^n$, compute $x^{(k)}$ for $k = 0, 1, 2, \dots$ using the following iterative scheme until $\{x^{(k)}\}_{k=0}^\infty$ converges,

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b, \end{cases} \quad (5)$$

where α is a positive constant and I is the identity matrix.

In the matrix-vector form, the HSS iteration can be equivalently rewritten as

$$x^{(k+1)} = T(\alpha)x^{(k)} + G(\alpha)b = T(\alpha)^{k+1}x^{(0)} + \sum_{j=0}^k T(\alpha)^j G(\alpha)b, \quad k = 0, 1, 2, \dots,$$

where

$$T(\alpha) = (\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S) \text{ and } G(\alpha) = 2\alpha(\alpha I + S)^{-1}(\alpha I + H)^{-1}.$$

Here, $T(\alpha)$ is the iterative matrix of the HSS method.

When the matrix $A \in \mathbb{C}^{n \times n}$ is positive definite, i.e. its Hermitian part $H = \frac{1}{2}(A + A^H)$ is positive definite, Bai et al proved that the spectral radius $\rho(T(\alpha))$ of the HSS iterative matrix $T(\alpha)$ is less than 1 for any positive iterative parameters α , i.e., the HSS iteration method unconditionally converges to the exact solution of $Ax = b$ for any initial guess $x^{(0)} \in \mathbb{C}^n$; see [1].

2.2 The Picard-HSS iterative methods

Recalling that the Picard iterative method is a fixed-point iterative method and the linear term Ax and the nonlinear term $|x| + b$ are separated, the AVE can be solved by using of the Picard iterative method

$$Ax^{(k+1)} = |x^{(k)}| + b, \quad k = 0, 1, \dots$$

When we assume that the matrix $A \in \mathbb{C}^{n \times n}$ is a large sparse and positive definite, the next iterate $x^{(k+1)}$ may be inexactly computed by HSS iteration. Thus we obtain the following iteration method proposed in [16] for solving the AVE (1).

Algorithm 2 The Picard-HSS iteration method.

Let the matrix $A \in \mathbb{C}^{n \times n}$ be positive definite with $H = \frac{1}{2}(A + A^H)$ and $S = \frac{1}{2}(A - A^H)$ being the Hermitian and skew-Hermitian parts of A , respectively. Given an initial guess $x^{(0)} \in \mathbb{C}^n$ and a sequence $\{l_k\}_{k=0}^\infty$ of positive integers, compute $x^{(k+1)}$ for $k = 0, 1, 2, \dots$ using the following iteration scheme until $\{x^{(k)}\}$ satisfies the stopping criterion:

- (a) Set $x^{(k,0)} := x^{(k)}$;
- (b) For $l = 0, 1, \dots, l_k - 1$, solve the following linear systems to obtain $x^{(k,l+1)}$:

$$\begin{cases} (\alpha I + H)x^{(k,l+\frac{1}{2})} = (\alpha I - S)x^{(k,l)} + |x^{(k)}| + b, \\ (\alpha I + S)x^{(k,l+1)} = (\alpha I - H)x^{(k,l+\frac{1}{2})} + |x^{(k)}| + b, \end{cases} \quad (6)$$

where α is a given positive constant;

- (c) Set $x^{(k+1)} := x^{(k,l_k)}$.

The advantage of Picard-HSS iterative method is obvious. First, the two linear sub-systems in all inner HSS iterations have the same shifted Hermitian coefficient matrix $\alpha I + H$ and shifted skew-Hermitian coefficient matrix $\alpha I + S$, which are constant with respect to the iteration index k . Second, As the coefficient matrix $\alpha I + H$ and $\alpha I + S$ are Hermitian and skew-Hermitian, respectively, the first sub-system can be solved exactly by making use of the Cholesky factorization and the second one by the LU factorization of the coefficient matrix. The last, these sub-systems can be solve approximately by the conjugate gradient method and a Krylov subspace method like GMRES, respectively; see [2, 16].

3 The nonlinear HSS-like iterative method

In the Picard-HSS iteration, the numbers l_k , $k = 0, 1, 2, \dots$ of the inner HSS iterative steps are often problem-dependent and difficult to be determined in actual computations[2]. Moreover, the iterative vector can not be updated timely. Thus, to avoid the defect and still preserve the advantages of the Picard-HSS iterative method, based on the nonlinear fixed-point equations

$$(\alpha I + H)x = (\alpha I - S)x + |x| + b, \quad \text{and} \quad (\alpha I + S)x = (\alpha I - H)x + |x| + b,$$

we propose the following nonlinear HSS-like iteration method.

Algorithm 3 The nonlinear HSS-like iteration method.

Let the matrix $A \in \mathbb{C}^{n \times n}$ be positive definite with $H = \frac{1}{2}(A + A^H)$ and $S = \frac{1}{2}(A - A^H)$ being the Hermitian and skew-Hermitian parts of A , respectively. Given an initial guess $x^{(0)} \in \mathbb{C}^n$, compute $x^{(k)}$ for $k = 0, 1, 2, \dots$ using the following iteration scheme until $\{x^{(k)}\}$ satisfies the stopping criterion,

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(l)} + |x^{(k)}| + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + |x^{(k+\frac{1}{2})}| + b, \end{cases} \quad (7)$$

where α is a given positive constant.

It is obvious that both x and $|x|$ in the second step are updated timely in the nonlinear HSS-like iteration, but only x is updated in the Picard-HSS iteration. Furthermore, the HSS-like iteration is a monolayer iterative scheme, but the Picard-HSS is an inner-outer double-layer iterative scheme.

To obtain a one-step form of the nonlinear HSS-like iterative process, we define

$$\begin{cases} U(x) = (\alpha I + H)^{-1}((\alpha I - S)x + |x| + b), \\ V(x) = (\alpha I + S)^{-1}((\alpha I - H)x + |x| + b), \end{cases} \quad (8)$$

and

$$\psi(x) = V \circ U(x) := V(U(x)). \quad (9)$$

Then the nonlinear HSS-like iterative scheme can be equivalently expressed as

$$x^{(k+1)} = \psi(x^{(k)}). \quad (10)$$

The Ostrowski theorem, i.e., Theorem 10.1.3 in [11], gives a local convergence theory about a one-step stationary nonlinear iteration. Based on this, Bai established the local convergence theory for the nonlinear HSS-like iteration method in [2]. However, these convergence theory has a strict requirement that $\phi(x) = |x| + b$ is F-differentiable at a point $x^* \in \mathbb{D}$ such that $Ax^* - |x^*| = b$. Obviously, the absolute value function $|x|$ is nondifferentiable.

Leveraging the smoothing approximate function introduced in [17], we can establish the following local convergence theory for nonlinear HSS-like iterative method. But firstly, we must review this smoothing approximation and its properties, which will be used in the next section.

Define $\phi : \mathbb{D} \subset \mathbb{C}^n \rightarrow \mathbb{C}^n$ by

$$\phi(x) = \frac{1}{\mu} \ln \left(\exp\left(\frac{x}{\mu}\right) + \exp\left(\frac{-x}{\mu}\right) \right), \quad x \in \mathbb{D}. \quad (11)$$

It is clear that $\phi(x)$ is a smoothing function of $|x|$, now we give some properties of $\phi(x)$, which will be used in the following text.

Lemma 1 ([17]) $\phi(x)$ is a uniformly smoothing approximation function of $|x|$, i.e.,

$$\|\phi(x) - |x|\| \leq \sqrt{n} \ln 2 \cdot \mu. \quad (12)$$

Lemma 2 ([17]) For any $\mu > 0$, the Jacobian of $\phi(x)$ at $x \in \mathbb{C}^n$ is

$$D = \phi'(x) = \text{diag} \left(\frac{\exp(\frac{x_i}{\mu}) - \exp(\frac{-x_i}{\mu})}{\exp(\frac{x_i}{\mu}) + \exp(\frac{-x_i}{\mu})}, i = 1, 2, \dots, n \right). \quad (13)$$

Lemma 3 ([2]) Assume that $\phi(x) : \mathbb{D} \subset \mathbb{C}^n \rightarrow \mathbb{C}^n$ is F-differentiable at a point $x^* \in \mathbb{D}$ such that $Ax^* = \phi(x^*) + b$. Suppose $A = H + S$, where $H = \frac{1}{2}(A + A^H)$ and $S = \frac{1}{2}(A - A^H)$ are the Hermitian and the skew-Hermitian parts of the matrix A , respectively. Denote by

$$T(\alpha, x^*) = (\alpha I + S)^{-1}(\alpha I - H + \phi'(x^*))(\alpha I + H)(\alpha I - S + \phi'(x^*))$$

and

$$\delta = \max \{ \|\phi'(x^*)(\alpha I + S)^{-1}\|_2, \|\phi'(x^*)(\alpha I + H)^{-1}\|_2 \},$$

$$\theta(\alpha) = \|(\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)(\alpha I - S)\|.$$

Then $\rho(T(\alpha; x^*)) < 1$ holds, in other word, $x^* \in \mathbb{D} \subset \mathbb{C}^n$ is a point of attraction of the nonlinear HSS-like iteration, provided

$$\delta = \frac{2(1 - \theta(\alpha))}{1 + \theta(\alpha) + \sqrt{1 - \theta(\alpha)^2 + 4}}.$$

Leveraging the smoothing approximate function $\phi(x)$ in (11), we define

$$\bar{U}(x) = (\alpha I + H)^{-1}((\alpha I - S)x + \phi(x) + b),$$

$$\bar{V}(x) = (\alpha I + S)^{-1}((\alpha I - H)x + \phi(x) + b),$$

and

$$\bar{\psi}(x) = \bar{V} \circ \bar{U}(x) := \bar{V}(\bar{U}(x)).$$

Then we have the smoothing nonlinear HSS-like iterative scheme

$$\bar{x}^{(k+1)} = \bar{\psi}(x^{(k)}). \quad (14)$$

Theorem 1 Assume that the condition of [Theorem 1](#) are stisfied, $H = \frac{1}{2}(A + A^H)$ and $S = \frac{1}{2}(A - A^H)$ be Hermitian and skew-Hermitian parts of the matrix A , respectively. For any initial guess $x^{(0)} \in \mathbb{D} \subset \mathbb{C}^n$, the iteration sequence $\{x^{(k)}\}_{k=0}^\infty$ produced by the nonlinear HSS-like iteration (10) can be instead approximately by that produced by its smoothed scheme (14), i.e.,

$$\|\psi(x^k) - \bar{\psi}(x^k)\| \leq \varepsilon, \quad \text{for } \forall \varepsilon > 0,$$

provided

$$\mu \leq \frac{\|(\alpha I + S)\|}{3\sqrt{n} \ln 2} \varepsilon.$$

Proof. Based on iterative scheme (10) and (14), we have

$$\begin{aligned} \|\bar{x}^{(k+1)} - x^{(k+1)}\| &= \|\bar{\psi}(x^{(k)}) - \psi(x^{(k)})\| \\ &\leq \|(\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\phi(x) - |x|)\| \\ &\quad + \|(\alpha I + S)^{-1}(\phi(\bar{U}(x)) - |U(x)|)\| \\ &\leq \|(\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}\| \|\phi(x) - |x|\| \\ &\quad + \|(\alpha I + S)^{-1}(\phi(\bar{U}(x)) - |\bar{U}(x)| + |\bar{U}(x)| - |U(x)|)\| \\ &\leq \|(\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}\| \|\phi(x) - |x|\| \\ &\quad + \|(\alpha I + S)^{-1}\| \cdot \|(\phi(\bar{U}(x)) - |\bar{U}(x)|)\| \\ &\quad + \|(\alpha I + S)^{-1}\| \cdot \|(\alpha I + H)^{-1}\| \cdot \|\phi(x) - |x|\| \\ &\leq \|(\alpha I + S)^{-1}\| 3\sqrt{n} \ln 2 \cdot \mu \\ &\leq \frac{3\sqrt{n} \ln 2}{\|(\alpha I + S)\|} \cdot \mu. \end{aligned}$$

For $\forall \varepsilon$, $\|\bar{x}^{(k+1)} - x^{(k+1)}\| = \|\psi(x^k) - \bar{\psi}(x^k)\| \leq \varepsilon$ holds, provided

$$\mu \leq \frac{\|(\alpha I + S)\|}{3\sqrt{n} \ln 2} \varepsilon.$$

□

Theorem 2 Assume that the condition of [Theorem 1](#) are satisfied. Denoted by

$$\delta = \max\{\|(\alpha I + H)^{-1}\|_2, \|(\alpha I + S)^{-1}\|_2\}$$

and

$$\theta(\alpha) = \|T(\alpha)\|_2 = \|(\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)\|_2.$$

Then the spectral radius $\rho(T(\alpha, x^*))$ of the matrix $T(\alpha, x^*)$ is less than 1, where

$$T(\alpha, x^*) = (\alpha I + S)^{-1}(\alpha I - H + D)(\alpha I + H)(\alpha I - S + D),$$

and D is the the Jacobian of $\phi(x)$ at $x^* \in \mathbb{N}(x^*) \subset \mathbb{D} \subset \mathbb{C}^n$ defined in (13), provided that

$$\delta < \frac{2(1 - \theta(\alpha))}{1 + \theta(\alpha) + \sqrt{(1 - \theta(\alpha))^2 + 4}}. \quad (15)$$

That is to say, for any initial guess $x^{(0)} \in \mathbb{D} \subset \mathbb{C}^n$, the iteration sequence $\{x^{(k)}\}_{k=0}^\infty$ produced by the nonlinear HSS-like iteration method converges to x^* , or x^* is a point of attraction of the ninlinear HSS-like iteration, provided the condition (15).

Proof : For $\forall \varepsilon$, We only need to prove

$$\|x^{k+1} - x^*\| \leq \|x^{k+1} - \bar{x}^{k+1}\| + \|\bar{x}^{k+1} - x^*\| \leq \|\psi(x^k) - \bar{\psi}(x^k)\| + \|\bar{\psi}(x^k) - x^*\| \leq \varepsilon, \quad (16)$$

where $\psi(x^k)$ is defined in (10) and $\bar{\psi}(x^k)$ is defined in (14).

According to the Theorem 1, the former part $\|\psi(x^k) - \bar{\psi}(x^k)\| \leq \varepsilon$ holds for $\forall \varepsilon$, provided

$$\mu \leq \frac{\|(\alpha I + S)\|}{3\sqrt{n} \ln 2} \varepsilon.$$

As the uniformly smoothing approximation function $\phi(x)$ of $|x|$ is F-differentiable at a point $x^* \in \mathbb{D}$ such that $Ax^* - |x^*| = b$, according Lemma 3, x^* is a point of attraction of the nonlinear HSS-like iteration, that is the second part in (16)

$$\|\bar{x}^{(k+1)} - x^*\| = \|\bar{\psi}(x^k) - x^*\| \leq \varepsilon$$

holds for $\forall \varepsilon$, provided $\rho(T(\alpha; x^*)) < 1$.

Next we prove $\rho(T(\alpha; x^*)) < 1$. By straightforward computations we have

$$\begin{aligned} (\alpha I + S)T(\alpha; x^*)(\alpha I + S)^{-1} &= (\alpha I + S)T(\alpha)(\alpha I + S)^{-1} \\ &\quad + (\alpha I - H)(\alpha I + H)^{-1}D(\alpha I + S)^{-1} \\ &\quad + D(\alpha I + H)^{-1}(\alpha I - S)(\alpha I + S)^{-1} \\ &\quad + D(\alpha I + H)^{-1}D(\alpha I + S)^{-1}, \end{aligned}$$

where D is the Jacobian of the smoothing approximation function $\phi(x)$ at $x^* \in \mathbb{N}(x^*) \subset \mathbb{D}$,

$$T(\alpha) = (\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S).$$

As $Q(\alpha) = (\alpha I - S)(\alpha I + S)^{-1}$ is a unitary transform, $\|D\|_2 \leq 1$ and

$$\begin{aligned} \theta(\alpha) &= \|T(\alpha)\|_2 \leq \|(\alpha I - S)(\alpha I + S)^{-1}\|_2 \|(\alpha I - H)(\alpha I + H)^{-1}\|_2 \\ &\leq \max_{\lambda_i \in \lambda(H)} \left| \frac{\alpha - \lambda_i}{\alpha + \lambda_i} \right| < 1, \end{aligned}$$

we obtain

$$\begin{aligned} \|T(\alpha; x^*)\|_2 &= \|(\alpha I + S)T(\alpha; x^*)(\alpha I + S)^{-1}\|_2 \\ &\leq \|(\alpha I + S)T(\alpha)(\alpha I + S)^{-1}\|_2 + \|(\alpha I - H)(\alpha I + H)^{-1}D(\alpha I + S)^{-1}\|_2 \\ &\quad + \|D(\alpha I + H)^{-1}(\alpha I - S)(\alpha I + S)^{-1}\|_2 + \|D(\alpha I + H)^{-1}D(\alpha I + S)^{-1}\|_2 \\ &\leq \|T(\alpha)\|_2 + \|(\alpha I - H)(\alpha I + H)^{-1}\|_2 \cdot \|D(\alpha I + S)^{-1}\|_2 \\ &\quad + \|D(\alpha I + H)^{-1}\|_2 \cdot \|(\alpha I - S)(\alpha I + S)^{-1}\|_2 \\ &\quad + \|D(\alpha I + H)^{-1}\|_2 \cdot \|D(\alpha I + S)^{-1}\|_2 \\ &\leq \|T(\alpha)\|_2 + \|(\alpha I - H)(\alpha I + H)^{-1}\|_2 \cdot \|(\alpha I + S)^{-1}\|_2 + \|(\alpha I + H)^{-1}\|_2 \\ &\quad \cdot \|(\alpha I + S)^{-1}\|_2 \\ &\leq \theta(\alpha) + (\theta(\alpha) + 1)\delta + \delta^2. \end{aligned}$$

Now, under the condition (15), we have $\rho(T(\alpha; x^*)) \leq \|T(\alpha; x^*)\|_2 < 1$. \square

Remark 1 An attractive feature of the nonlinear HSS-like iterative method is that it avoids the use of the differentiable in actual iterative scheme, although we employ it in the convergence analysis. Thus, the smoothing approximate function $\phi(x)$ in (11) is not necessary in actual implementation.

At the end of this section, we remark that the main steps in HSS-like iteration method can be alternatively reformulated into residual-updating form similar to those in the Picard-HSS iterative method as follows.

Algorithm 4 (The HSS-like iteration method (residual-updating variant))

Let the matrix $A \in \mathbb{C}^{n \times n}$ be positive definite with $H = \frac{1}{2}(A + A^H)$ and $S = \frac{1}{2}(A - A^H)$ being the Hermitian and skew-Hermitian parts of A , respectively. Given an initial guess $x^{(0)} \in \mathbb{D} \subset \mathbb{C}^n$, compute $x^{(k+1)}$ for $k = 0, 1, 2, \dots$ using the following iteration procedure until $\{x^{(k)}\}$ satisfies the following stopping criterion:

$$\begin{cases} r^{(k)} := |x^{(k)}| + b - Ax^{(k)}, \\ (\alpha I + H)v = r^{(k)}, \\ x^{(k+\frac{1}{2})} = x^{(k)} + v, \\ r^{(k)} := |x^{(k+\frac{1}{2})}| + b - Ax^{(k+\frac{1}{2})}, \\ (\alpha I + S)v = r^{(k)}, \\ x^{(k+1)} = x^{(k+\frac{1}{2})} + v, \end{cases} \quad (17)$$

where α is a given positive constant.

4 Numerical experiments

In this section, the numerical properties of the Picard, Picard-HSS and nonlinear HSS-like methods are examined and compared experimentally by a suit of test problems. All the tests are performed in MATLAB R2013a on Intel(R) Core(TM) i5-3470 CPU 3.20 GHz and 8.00 GB of RAM, with machine precision 10^{-16} , and terminated when the current residual satisfies

$$\frac{\|Ax^{(k)} - |x^{(k)}| - b\|_2}{\|b\|_2} \leq 10^{-5}, \quad (18)$$

where $x^{(k)}$ is the computed solution by each of the methods at iterate k , and a maximum number of the iterations 500 is used.

In addition, the stopping criterion for the inner iterations of the Picard-HSS method are set to be

$$\frac{\|b^{(k)} - As^{(k,l_k)}\|_2}{\|b^{(k)}\|_2} \leq \eta_k, \quad (19)$$

where $b^{(k)} = |x^{(k)}| + b - Ax^{(k)}$, l_k is the number of the inner iteration steps and η_k is the prescribed tolerance for controlling the accuracy of the inner iterations at the k -th outer iterate. If η_k is fixed for all k , then it is simply denoted by η .

In our numerical experiments, we use the zero vector as the initial guess, the accuracy of the inner iterations η_k for Picard-HSS iterative method is fixed and set to 0.1, and the right-hand side vector b of AVE (1) is taken in such a way that the vector $x = (x_1, x_2, \dots, x_n)^T$ with

$$x_k = (-1)^k \mathbf{i}, \quad k = 1, 2, \dots, n, \quad (20)$$

be the exact solution. The first subsystems with the Hermitian positive definite coefficient matrix $(\alpha I + H)$ in (7) are solved by the Cholesky factorization of the coefficient matrix and the second subsystems with the skew-Hermitian coefficient matrix $(\alpha I + S)$ in (7) are solved by the LU factorization of the coefficient matrix.

The optimal parameters employed in the Picard-HSS and nonlinear HSS-like iterative method have been obtained experimentally. In fact, the experimentally found optimal parameters α are the ones resulting in the least numbers of iterations and CPU times[16]. As mentioned in [2] the computation of the optimal parameter is often problem-dependent and generally difficult to be determined.

For our numerical experiments, we consider the two-dimensional convection-diffusion equation

$$\begin{cases} -(u_{xx} + u_{yy}) + q(u_x + u_y) + pu = f(x, y), & (x, y) \in \Omega, \\ u(x, y) = 0, & (x, y) \in \partial\Omega, \end{cases} \quad (21)$$

where $\Omega = (0, 1) \times (0, 1)$, $\partial\Omega$ its boundary, q is a positive constant used to measure the magnitude of the diffusive term and p is a real number. We use the five-point finite difference scheme to the diffusive terms and the central difference scheme to the convective terms. Let $h = 1/(m + 1)$ and $Re = (qh)/2$ denote the equidistant step size and the mesh Reynolds number, respectively. Then we get a system of linear equations $Bx = d$, where B is matrix of order $n = m^2$ of the form

$$B = T_x \otimes I_m + I_m \otimes T_y + pI_n, \quad (22)$$

wherein I_m and I_n are, respectively, the identity matrices of order m and n , \otimes means the Kronecker product symbol, and T_x and T_y are the tridiagonal matrices

$$T_x = \text{tridiag}(t_2, t_1, t_3)_{m \times m} \quad \text{and} \quad T_y = \text{tridiag}(t_2, 0, t_3)_{m \times m}, \quad (23)$$

with

$$t_1 = 4, \quad t_2 = -1 - Re, \quad t_3 = -1 + Re. \quad (24)$$

It is easy to see that for every nonnegative number q the matrix B is in general non-symmetric positive definite[16].

By making use of the matrix B for our numerical experiments, We define the matrix $A = B$ in AVE (1), B is defined by (22) with different values of q ($q = 0, 1, 10, 100$ and 1000) and different values of p ($p = 0$ and 0.5).

In Table 3 and Table 4, we present the numerical results with respect to the Picard, Picard-HSS and nonlinear HSS-like iteration, the experimentally optimal parameters used in the Picard-HSS and nonlinear HSS-like iteration are those given in Table 1 and Table 2. we give the elapsed CPU time in seconds for the convergence (denoted as CPU), the norm of absolute residual vectors (denoted as RES), and the number of outer, inner and total iteration steps (outer and inner iterations only for Picard-HSS) for the convergence (denoted as IT_{out} , IT_{int} and IT , respectively). The number of outer iterative steps for Picard-HSS and the number of iterative steps for Picard and HSS-like iterative methods larger than 500 are simply listed by the symbol "-".

From these two tables, we see that both the HSS-like and Picard-HSS methods can successfully produced approximate solutions to the AVE, for all of the problem-scales $n = m^2$ and the convective measurements q , while Picard iteration converge only for some special

Table 1 The optimal parameters values α for Picard-HSS and nonlinear HSS-like methods ($p=0$).

| Optimal parameters | | m=10 | m=20 | m=40 | m=80 |
|--------------------|------------|------|------|------|------|
| q=0 | HSS-like | 1.3 | 1.0 | 1.0 | 1.0 |
| | Picard-HSS | 1.1 | 0.5 | 0.2 | 0.1 |
| q=1 | HSS-like | 1.4 | 1.0 | 1.0 | 1.0 |
| | Picard-HSS | 1.1 | 0.6 | 0.3 | 0.2 |
| q=10 | HSS-like | 1.7 | 1.1 | 1.0 | 1.0 |
| | Picard-HSS | 1.6 | 0.8 | 0.4 | 0.2 |
| q=100 | HSS-like | 2.5 | 2.7 | 1.7 | 1.2 |
| | Picard-HSS | 2.4 | 2.7 | 1.8 | 0.9 |
| q=1000 | HSS-like | 1.9 | 1.1 | 2.9 | 2.3 |
| | Picard-HSS | 1.9 | 1.1 | 2.9 | 2.3 |

Table 2 The optimal parameters values α for Picard-HSS and nonlinear HSS-like methods ($p=0.5$).

| Optimal parameters | | m=10 | m=20 | m=40 | m=80 |
|--------------------|------------|------|------|------|------|
| q=0 | HSS-like | 2.4 | 2.2 | 2.1 | 2.0 |
| | Picard-HSS | 2.2 | 2.0 | 1.8 | 1.8 |
| q=1 | HSS-like | 2.4 | 2.2 | 2.1 | 2.0 |
| | Picard-HSS | 2.3 | 2.0 | 1.8 | 1.8 |
| q=10 | HSS-like | 2.6 | 2.3 | 2.2 | 2.1 |
| | Picard-HSS | 2.4 | 2.3 | 2.0 | 1.9 |
| q=100 | HSS-like | 3.4 | 2.9 | 2.3 | 2.3 |
| | Picard-HSS | 3.5 | 3.0 | 2.3 | 2.1 |
| q=1000 | HSS-like | 2.9 | 2.4 | 2.4 | 2.5 |
| | Picard-HSS | 2.8 | 2.4 | 2.4 | 2.5 |

cases. Here, it is necessary to mention that the shifted matrices $\alpha I + H$ and $\alpha I + S$ are usually more well-conditioned than the matrix A [16].

For the convergent cases, the number of iteration steps for Picard and HSS-like and the number of inner iteration steps for Picard-HSS are increase rapidly with the increasing of problem scale, while the number of outer iteration steps is fixed. The CPU time also increases rapidly with the increasing of the problem scale for all iterative methods.

When the convective measurements q become large, for all iterative method, both the number of iteration steps (except outer iteration for Picard-HSS) and the amount of CPU times decrease, while $q = 1000$ is in the opposite situation.

Clearly, in terms of iteration step, the nonlinear HSS-like method and the Picard-HSS are more robust than Picard, and the nonlinear HSS-like method performs much better than the Picard-HSS; In terms of CPU time, the situation is almost the same, but the Picard iterative method is the most time-efficient in the convergent cases, e.g. $q = 1000$. Therefore, the nonlinear HSS-like method are the winners for solving this test problem when q is small.

5 Conclusions

In this paper we have studied the nonlinear HSS-like for absolute value equation (AVE). The nonlinear HSS-like iterative method is based on separable property of the linear term Ax and nonlinear term $|x| + b$ and the Hermitian and skew-Hermitian splitting of involved matrix

A. Compared to the Picard-HSS iterative scheme is an inner-outer double-layer iterative scheme, the nonlinear HSS-like iteration is a monolayer and the iterative vector could be updated timely. By leveraging the smoothing approximate function, the locally convergence have been analysed. Further numerical experiments have shown that the nonlinear HSS-like method is feasible, robust and efficient nonlinear solver. The most important is it can outperform the Picard-HSS in actual implementation.

A defect of the nonlinear HSS-like iterative method designed for solving weakly nonlinear systems to solve AVE is that the smoothing approximate function is introduced in the convergence analysis, although it is avoid in practical algorithm. Hence, to find a better theoretical proof for HSS-like will be a topics in the future research.

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Table 3 Numerical results for test problems with different values of m and q ($p = 0$, RES (10^{-6})).

| Methods | | | m=10 | m=20 | m=40 | m=80 |
|---------|------------|-------------------|--------|--------|--------|--------|
| q=0 | HSS-like | IT | 27 | 35 | 65 | 81 |
| | | CPU | 0.0375 | 0.0146 | 0.1016 | 0.6085 |
| | | RES | 9.4084 | 8.7487 | 9.9395 | 9.9502 |
| | Picard-HSS | IT _{out} | 5 | 5 | 5 | 5 |
| | | IT _{int} | 7.2 | 13.8 | 33 | 62.6 |
| | | IT | 36 | 69 | 165 | 313 |
| | Picard | CPU | 0.0084 | 0.0250 | 0.2310 | 2.0708 |
| | | RES | 5.2907 | 7.1401 | 7.9627 | 9.1458 |
| | | IT | – | – | – | – |
| | Picard | CPU | – | – | – | – |
| | | RES | – | – | – | – |
| q=1 | HSS-like | IT | 28 | 38 | 65 | 81 |
| | | CPU | 0.0044 | 0.0199 | 0.1343 | 0.8436 |
| | | RES | 8.7445 | 9.5272 | 9.9148 | 9.9588 |
| | Picard-HSS | IT _{out} | 5 | 5 | 5 | 5 |
| | | IT _{int} | 7.2 | 13.6 | 27 | 64.8 |
| | | IT | 36 | 68 | 135 | 324 |
| | Picard | CPU | 0.0050 | 0.0317 | 0.2527 | 3.0404 |
| | | RES | 6.3073 | 8.0703 | 7.7121 | 9.3360 |
| | | IT | – | – | – | – |
| | Picard | CPU | – | – | – | – |
| | | RES | – | – | – | – |
| q=10 | HSS-like | IT | 17 | 32 | 51 | 85 |
| | | CPU | 0.0029 | 0.0176 | 0.1077 | 0.8857 |
| | | RES | 7.8979 | 7.2166 | 9.3825 | 9.8324 |
| | Picard-HSS | IT _{out} | 5 | 5 | 5 | 5 |
| | | IT _{int} | 3.8 | 7 | 13.2 | 25.4 |
| | | IT | 19 | 35 | 66 | 127 |
| | Picard | CPU | 0.0031 | 0.0174 | 0.1285 | 1.2305 |
| | | RES | 2.6888 | 4.0994 | 5.9529 | 7.1369 |
| | | IT | – | – | – | – |
| | Picard | CPU | – | – | – | – |
| | | RES | – | – | – | – |
| q=100 | HSS-like | IT | 18 | 20 | 25 | 42 |
| | | CPU | 0.0037 | 0.0117 | 0.0574 | 0.4687 |
| | | RES | 8.2690 | 8.8682 | 7.5469 | 9.3710 |
| | Picard-HSS | IT _{out} | 5 | 5 | 5 | 5 |
| | | IT _{int} | 3.8 | 4.2 | 5.6 | 8.2 |
| | | IT | 19 | 21 | 28 | 41 |
| | Picard | CPU | 0.0039 | 0.0116 | 0.0602 | 0.4413 |
| | | RES | 3.2385 | 3.3042 | 3.3640 | 3.6666 |
| | | IT | 4 | 8 | 39 | – |
| | Picard | CPU | 0.0009 | 0.0036 | 0.0436 | – |
| | | RES | 6.9831 | 0.0032 | 6.8249 | – |
| q=1000 | HSS-like | IT | 21 | 38 | 50 | 51 |
| | | CPU | 0.0045 | 0.0330 | 0.2909 | 2.3601 |
| | | RES | 9.3457 | 8.6466 | 8.4651 | 8.9837 |
| | Picard-HSS | IT _{out} | 5 | 5 | 5 | 5 |
| | | IT _{int} | 5 | 7.6 | 10 | 10 |
| | | IT | 25 | 38 | 50 | 50 |
| | Picard | CPU | 0.0049 | 0.0318 | 0.2872 | 2.3036 |
| | | RES | 0.8263 | 3.4996 | 6.6572 | 8.5153 |
| | | IT | 3 | 4 | 5 | 12 |
| | Picard | CPU | 0.0009 | 0.0047 | 0.0442 | 0.7562 |
| | | RES | 1.4196 | 0.0006 | 0.0338 | 0.0005 |

Table 4 Numerical results for test problems with different values of m and q ($p = 0$, RES (10^{-6})).

| Methods | | | m=10 | m=20 | m=40 | m=80 |
|---------|------------|-------------------|--------|--------|--------|--------|
| q=0 | HSS-like | IT | 29 | 38 | 36 | 35 |
| | | CPU | 0.0037 | 0.0155 | 0.0590 | 0.2849 |
| | | RES | 7.7828 | 8.0756 | 9.6565 | 8.8724 |
| | Picard-HSS | IT _{out} | 5 | 5 | 5 | 5 |
| | | IT _{int} | 7 | 14.6 | 35 | 66.4 |
| | | IT | 35 | 73 | 175 | 332 |
| | Picard | CPU | 0.0040 | 0.0261 | 0.2420 | 2.2039 |
| | | RES | 5.4444 | 7.4483 | 8.1466 | 9.3423 |
| | | IT | 9 | – | – | – |
| | | CPU | 0.0010 | – | – | – |
| | | RES | 0.0016 | – | – | – |
| q=1 | HSS-like | IT | 29 | 42 | 38 | 36 |
| | | CPU | 0.0044 | 0.0218 | 0.0824 | 0.4113 |
| | | RES | 8.1442 | 8.5129 | 9.8553 | 8.2976 |
| | Picard-HSS | IT _{out} | 5 | 5 | 5 | 5 |
| | | IT _{int} | 7.8 | 14.4 | 28 | 42 |
| | | IT | 39 | 72 | 140 | 210 |
| | Picard | CPU | 0.0052 | 0.0335 | 0.2612 | 1.9946 |
| | | RES | 4.2330 | 5.3548 | 8.8367 | 8.5786 |
| | | IT | 9 | – | – | – |
| | | CPU | 0.0011 | – | – | – |
| | | RES | 0.0011 | – | – | – |
| q=10 | HSS-like | IT | 18 | 34 | 45 | 42 |
| | | CPU | 0.0030 | 0.0183 | 0.0960 | 0.4848 |
| | | RES | 8.1728 | 6.0961 | 8.8821 | 9.2731 |
| | Picard-HSS | IT _{out} | 5 | 5 | 5 | 5 |
| | | IT _{int} | 4 | 7 | 13.6 | 25 |
| | | IT | 20 | 35 | 68 | 125 |
| | Picard | CPU | 0.0032 | 0.0179 | 0.1379 | 1.2244 |
| | | RES | 1.5905 | 5.9853 | 5.1449 | 8.9996 |
| | | IT | 7 | – | – | – |
| | | CPU | 0.0009 | – | – | – |
| | | RES | 0.1525 | – | – | – |
| q=100 | HSS-like | IT | 14 | 14 | 22 | 37 |
| | | CPU | 0.0032 | 0.0088 | 0.0518 | 0.4204 |
| | | RES | 9.8625 | 5.9430 | 5.6508 | 7.4515 |
| | Picard-HSS | IT _{out} | 5 | 5 | 5 | 5 |
| | | IT _{int} | 3.4 | 3.2 | 5.5 | 8.4 |
| | | IT | 17 | 16 | 22 | 42 |
| | Picard | CPU | 0.0037 | 0.0093 | 0.0498 | 0.4558 |
| | | RES | 1.4643 | 1.2321 | 2.7830 | 4.4858 |
| | | IT | 4 | 6 | 14 | – |
| | | CPU | 0.0009 | 0.0030 | 0.0205 | – |
| | | RES | 0.9480 | 0.0162 | 0.0229 | – |
| q=1000 | HSS-like | IT | 21 | 38 | 19 | 18 |
| | | CPU | 0.0044 | 0.0330 | 0.1312 | 1.1059 |
| | | RES | 7.9169 | 8.2226 | 8.5076 | 6.6481 |
| | Picard-HSS | IT _{out} | 5 | 5 | 5 | 5 |
| | | IT _{int} | 4.6 | 7.6 | 4.4 | 4 |
| | | IT | 23 | 38 | 22 | 20 |
| | Picard | CPU | 0.0047 | 0.0322 | 0.1452 | 1.1742 |
| | | RES | 2.2895 | 5.6813 | 1.4223 | 2.0359 |
| | | IT | 3 | 4 | 5 | 7 |
| | | CPU | 0.0009 | 0.0046 | 0.0419 | 0.6358 |
| | | RES | 0.7292 | 0.0002 | 0.0003 | 0.3046 |

